Resit Exam — Complex Analysis

Aletta Jacobshal 01, Wednesday 25 February 2015, 18:30 - 21:30

Duration: 3 hours

Instructions

1. The test consists of 6 questions; answer all of them.
2. Each question gets 15 points and the number of points for each subquestion is indicated at the beginning of the subquestion. 10 points are "free" and the total number of points is divided by 10. The final grade will be between 1 and 10.
3. The use of books, notes, and calculators is not allowed.

Question 1 (15 points)

Consider the function $f(z) = ze^{-iz}$ with z in $\mathbb{C}$.

a. (7 points) Write $f(z)$ as a sum of a real and an imaginary part, in other words, in the form $u(x, y) + iv(x, y)$ where $z = x + iy$.

b. (8 points) Use the Cauchy-Riemann equations to show that $f(z)$ is entire.

Solution

a. We have

$$\begin{aligned} f(z) &= (x+iy)e^{-i(x+iy)} = (x+iy)e^{y-ix} = (x+iy)e^{y}(\cos x - i\sin x) \\ &= e^{y}(x\cos x + y\sin x) + ie^{y}(y\cos x - x\sin x). \end{aligned}$$

Therefore

$$u(x, y) = e^{y}(x\cos x + y\sin x), \quad v(x, y) = e^{y}(y\cos x - x\sin x).$$

b. We check that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We have

$$\frac{\partial u}{\partial x} = e^{y}(\cos x - x\sin x + y\cos x),$$

and

$$\frac{\partial v}{\partial y} = e^{y}(y\cos x - x\sin x + \cos x) = \frac{\partial u}{\partial x}.$$

Furthermore,

$$\frac{\partial u}{\partial y} = e^{y}(x\cos x + y\sin x + \sin x),$$

and

$$\frac{\partial v}{\partial x} = e^{y}(-y\sin x - \sin x - x\cos x) = -\frac{\partial u}{\partial y}.$$

Therefore the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous on $\mathbb{C}$ so f is analytic on $\mathbb{C}$, that is, entire.

Question 2 (15 points)

Consider the function

$$f(z) = \frac{1}{z(z+1)}.$$

a. (9 points) Find the Laurent series for $f(z)$ in $0 < |z| < 1$.

b. (6 points) What is the type of the singularity of $f(z)$ at 0? Explain your answer.

Solution

a. The Taylor series for $1/(1-w)$ is known to be

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots = \sum_{j=0}^{\infty} w^j,$$

for $|w| < 1$. Then

$$\frac{1}{z+1} = \frac{1}{1-(-z)} = \sum_{j=0}^{\infty} (-z)^j = 1 - z + z^2 - z^3 + \cdots ,$$

for $|-z| < 1$, i.e., for $|z| < 1$.
Therefore, for $|z| < 1$ we have

$$\frac{1}{z(z+1)} = \frac{1}{z}\sum_{j=0}^{\infty}(-z)^j = \sum_{j=0}^{\infty}(-1)^j z^{j-1} = \sum_{k=-1}^{\infty}(-1)^{k+1} z^k = \frac{1}{z} - 1 + z - z^2 + \cdots .$$

b. The Laurent series of f at 0 contains exactly one negative power z^{-1} therefore 0 is a *pole of order* 1.

Question 3 (15 points)

Consider the function

$$f(z) = \frac{e^{iz}}{z^2+4}.$$

a. (6 points) Compute the residue of $f(z)$ at each one of the singularities of the function.

b. (9 points) Evaluate

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\, dx.$$

Solution

a. The given function

$$f(z) = \frac{e^{iz}}{z^2+4} = \frac{e^{iz}}{(z-2i)(z+2i)}.$$

has singularities at $z = \pm 2i$. We have

$$\text{Res}(f; 2i) = \lim_{z\to 2i} (z-2i) f(z) = \lim_{z\to 2i} \frac{e^{iz}}{z+2i} = \frac{1}{4ie^2},$$

and

$$\text{Res}(f; -2i) = \lim_{z\to -2i} (z+2i) f(z) = \lim_{z\to -2i} \frac{e^{iz}}{z-2i} = -\frac{e^2}{4i}.$$

b. We have

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\, dx = \lim_{r\to\infty} I_r,$$

where

$$I_r = \int_{-r}^{r} \frac{e^{ix}}{x^2+4}\, dx = \int_{\gamma_r} f(z)\, dz,$$

with $f(z)$ as in the previous subquestion and γ_r the straight line contour from $-r$ to r along the real axis.
We define the closed contour $\Gamma_r = \gamma_r + C_r^+$ where C_r^+ is the semicircle $|z| = r$ in the upper half-plane going from r to $-r$. Therefore

$$\int_{\Gamma_r} f(z)\, dz = I_r + \int_{C_r^+} f(z)\, dz.$$

This implies

$$\lim_{r\to\infty} \int_{\Gamma_r} f(z)\, dz = \lim_{r\to\infty} I_r + \lim_{r\to\infty} \int_{C_r^+} f(z)\, dz = I + \lim_{r\to\infty} \int_{C_r^+} f(z)\, dz,$$

and

$$I = \lim_{r\to\infty} \int_{\Gamma_r} f(z)\, dz - \lim_{r\to\infty} \int_{C_r^+} f(z)\, dz.$$

For any $r > 2$ we have that $2i$ is the only singularity of f inside Γ_r, therefore

$$\lim_{r\to\infty} \int_{\Gamma_r} f(z)\, dz = 2\pi i \operatorname{Res}(f, 2i) = 2\pi i \frac{1}{4ie^2} = \frac{\pi}{2e^2}.$$

From Jordan's lemma we also know that

$$\lim_{r\to\infty} \int_{C_r^+} \frac{e^{iz}}{z^2+4}\, dz = 0.$$

Therefore

$$I = \frac{\pi}{2e^2}.$$

Question 4 (15 points)

a. (7 points) Given the function

$$f(z) = z\,(z+2)\left(z-\frac{i}{2}\right)^2$$

compute the integral

$$\int_C \frac{f'(z)}{f(z)}\,dz,$$

where C is the positively oriented circular contour with $|z|=1$.

b. (8 points) Use Rouché's theorem to show that the polynomial $P(z) = z^3 - \frac{1}{2}z^2 + 1$ has exactly 3 roots in the disk $|z| < 2$.

Solution

a. The function f is entire and is therefore analytic on and inside C. We can then apply the Argument Principle to obtain that

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i N_0(f),$$

where $N_0(f)$ is the number of zeros of f inside C (counting multiplicities). The only zeros of f inside C are $i/2$ with multiplicity 2, and 0 with multiplicity 1. Therefore $N_0(f) = 3$ and

$$\int_C \frac{f'(z)}{f(z)}\,dz = 6\pi i.$$

b. Consider the closed contour C given by $|z| = 2$ and the functions $f(z) = z^3$ and $h(z) = -\frac{1}{2}z^2 + 1$, so that $P(z) = f(z) + h(z)$.
Then on C we have

$$|f(z)| = |z|^3 = 2^3 = 8,$$

and

$$|h(z)| \le \left|-\frac{1}{2}\right| |z|^2 + |1| = 3 < |f(z)|.$$

Both f and h are entire functions and in particular they are analytic on and inside C, and since $|f(z)| < h(z)$ on C, Rouché's theorem can be applied and we obtain

$$N_0(f) = N_0(P).$$

Furthermore, $N_0(f) = 3$ so $P(z)$ has 3 zeros inside C.

Question 5 (15 points)

We denote by $\operatorname{Log} z$ the principal value of the logarithm $\log z$.

a. (7 points) Prove that $\operatorname{Log}(e^z) = z$ *if and only if* $-\pi < \operatorname{Im} z \leq \pi$.

b. (8 points) Construct a branch of $\log(z+4)$ that is analytic at the point $z = -5$ and takes the value $7\pi i$ there.

Solution

a. We have that

$$\operatorname{Log} z = \operatorname{Log}|z| + i \operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the principal argument. Therefore, if we write $z = x + iy$ we have that

$$\operatorname{Log} e^z = \operatorname{Log}(e^x e^{iy}) = \operatorname{Log}(e^x) + i(y + 2k\pi) = x + i(y + 2k\pi),$$

where $k \in \mathbb{Z}$ is such that $-\pi < y + 2k\pi \leq \pi$.
Therefore, if

$$\operatorname{Log}(e^z) = z,$$

then

$$x + i(y + 2k\pi) = x + iy,$$

so $k = 0$, implying that $-\pi < y \leq \pi$. On the other hand, if $-\pi < y \leq \pi$ then k must be 0, therefore $\operatorname{Log}(e^z) = z$.
[One can also say that $\operatorname{Log}(e^z) = z$ if and only if $x + i(y + 2k\pi) = x + iy$ if and only if $k = 0$ if and only if $-\pi < y \leq \pi$.]

b. By letting $w = z + 4$ we can translate the given requirements to finding a branch of $\log(w)$ that is analytic at $w = -5 + 4 = -1$ and such that $\log(-1) = 7\pi i$.
Since we need that the branch is analytic at -1 we can make a cut along the positive x-axis (including 0). Note that

$$\log(-1) = \operatorname{Log}|-1| + i \arg(-1) = i \arg(-1),$$

so we need to take a branch of the argument that will give the value $\arg(-1) = 7\pi$.
In order to attain the required value we take the branch

$$\mathcal{L}_{6\pi}(w) = \operatorname{Log}|w| + i \arg_{6\pi}(w).$$

Then $6\pi < \arg_{6\pi}(w) \leq 8\pi$, so $\arg_{6\pi}(-1) = \pi + 2k\pi = 7\pi$ as required.
In conclusion, the required branch is

$$\mathcal{L}_{6\pi}(z+4) = \operatorname{Log}|z+4| + i \arg_{6\pi}(z+4).$$

Question 6 (15 points)

The generalized Cauchy integral formula gives that if $f(z)$ is analytic inside and on a circle C_r of radius r centered at z_0 then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

a. (9 points) Prove that if $f(z)$ is analytic *inside and on* a circle C_r of radius r centered at z_0 and if $|f(z)| \leq M$ for all z on C_r, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

b. (6 points) Prove that if $f(z)$ is analytic for all z in the domain $|z-z_0| < R$ and if $|f(z)| \leq M$ in the same domain, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

[Hint: Apply the result of the previous subquestion. Nevertheless, be careful that you cannot directly apply this result with $r = R$ since $f(z)$ does not have to be analytic on the circle C_R with $|z - z_0| = R$.]

Solution

a. From the generalized Cauchy integral formula we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

On C_r we have

$$\left| \frac{f(z)}{(z-z_0)^{n+1}} \right| = \frac{|f(z)|}{|z-z_0|^{n+1}} = \frac{|f(z)|}{r^{n+1}} \leq \frac{M}{r^{n+1}}.$$

Therefore,

$$|f^{(n)}(z_0)| \leq \frac{n!}{|2\pi i|} \frac{M}{r^{n+1}} (2\pi r) = \frac{n!M}{r^n}.$$

b. For any r with $0 < r < R$ we have from the given assumptions that $f(z)$ is analytic inside and on C_r and $|f(z)| \leq M$ on C_r. Therefore, for any such r we have from the previous subquestion

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

Suppose now that

$$|f^{(n)}(z_0)| > \frac{n!M}{R^n}.$$

Then

$$R > \left(\frac{n!M}{|f^{(n)}(z_0)|} \right)^{1/n}.$$

Take now any r such that

$$\left(\frac{n!M}{|f^{(n)}(z_0)|}\right)^{1/n} < r < R.$$

Then

$$\frac{n!M}{r^n} < |f^{(n)}(z_0)|$$

which contradicts our previous result. Therefore we must have

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

Another way to obtain the same result is to take the limit $r \to R^-$ of the relation

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

Then

$$|f^{(n)}(z_0)| \leq \lim_{r \to R^-} \frac{n!M}{r^n} = \frac{n!M}{R^n}.$$